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LETTER TO THE EDITOR

New coherent states of the Lie superalgebra $osp(1/2, R)$

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Abstract. A new kind of coherent states (CSS) for Lie superalgebra $osp(1/2, R)$ are introduced. Some properties of these states are discussed. It is shown that they can encompass the Glauber CSS, $su(1, 1)$ CSS and squeezed states within a common formalism. The D-algebra differential operator realization of the $osp(1/2, R)$ generators on the CSS and their diagonal projectors is constructed.

It is well known that coherent states (CSS) have been one of the most important elements in quantum physics since its early beginning. They have been applied to almost every area of physics and mathematical physics [1]. In the past few years, CSS for Lie superalgebras [2–5] have been introduced. In particular, the authors in [2, 3] discussed in detail CSS of the Lie superalgebra $osp(1/2, R)$ which is one of the most important and the simplest Lie superalgebras. They introduced $osp(1/2, R)$ CSS by using Perelomov's definition of CSS for arbitrary groups [6]. The CSS defined in this way have properties of being minimum uncertainty states in the sense that they minimize the dispersion of the quadratic Casimir operator of the algebra. The purpose of this letter is to present a new kind of CSS for the superalgebra $osp(1/2, R)$ and to construct a D-algebra differential operator realization of the $osp(1/2, R)$ generators on the CSS and their diagonal projectors.

The Lie superalgebra $osp(1/2, R)$ has five generators whose commutation and anticommutation relations are given by the following:

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0 \tag{1}$$

$$[K_0, F_\pm] = \pm\tfrac{1}{2}F_\pm \qquad [K_\pm, F_\pm] = 0 \qquad [K_\pm, F_\mp] = \mp F_\pm \tag{2}$$

$$\{F_\pm, F_\pm\} = K_\pm \qquad \{F_+, F_-\} = K_0 \tag{3}$$

which contains a subalgebra $su(1, 1)$ spanned by $K_\pm$ and K_0 as its even part. The Casimir operator of the superalgebra is

$$C_2 = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) + \tfrac{1}{2}(F_+F_- - F_-F_+). \tag{4}$$

The $osp(1/2, R)$ generators admit the following boson realization:

$$K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \qquad K_0 = \tfrac{1}{2}(a^\dagger a + \tfrac{1}{2}) \tag{5}$$

$$F_+ = \tfrac{1}{2}a^\dagger \qquad F_- = \tfrac{1}{2}a \tag{6}$$

0305-4470/94/050119+06$19.50

where $[a, a^\dagger] = 1, a$ and $a^\dagger$ are the annihilation and creation operators of a boson.

Making use of (5) and (6), one can obtain a number representation of the superalgebra:

$$K_+|n\rangle = \tfrac{1}{2}\sqrt{(n+1)(n+2)}|n+2\rangle \qquad K_-|n\rangle = \tfrac{1}{2}\sqrt{n(n-1)}|n-2\rangle \tag{7}$$

$$K_0|n\rangle = \tfrac{1}{2}(n+\tfrac{1}{2})|n\rangle \qquad F_+|n\rangle = \tfrac{1}{2}\sqrt{n+1}|n+1\rangle \qquad F_-|n\rangle = \tfrac{1}{2}\sqrt{n}|n-1\rangle. \tag{8}$$

For convenience, corresponding to even and odd parts of the $osp(1/2, R)$ algebra we first introduce the following two operators:

$$S(\beta) \equiv \exp(\beta K_+ - \beta^* K_-) \qquad D(\alpha) \equiv \exp(\alpha F_+ - \alpha^* F_-) \tag{9}$$

which satisfy the relation,

$$D(\alpha)S(\beta) = S(\beta)D(\alpha\cosh r + \alpha^* e^{i\theta}\sinh r) \qquad \beta = re^{i\theta}. \tag{10}$$

It can be shown that the operators $D(\alpha)$ and $S(\beta)$ act as a displacement operator and a rotation operator for the odd part of the superalgebra, respectively,

$$D^\dagger(\alpha)\begin{pmatrix} F_- \\ F_+ \end{pmatrix} D(\alpha) = \begin{pmatrix} F_- + \frac{1}{4}\alpha \\ F_+ + \frac{1}{4}\alpha^* \end{pmatrix} \tag{11}$$

$$S^\dagger(\beta)\begin{pmatrix} F_- \\ F_+ \end{pmatrix} S(\beta) = \begin{pmatrix} \cosh r & e^{i\theta}\sinh r \\ e^{-i\theta}\sinh r & \cosh r \end{pmatrix}\begin{pmatrix} F_- \\ F_+ \end{pmatrix}. \tag{12}$$

The actions of $D(\alpha)$ and $S(\beta)$ on generators K_- and K_+ are given by

$$D^\dagger(\alpha)\begin{pmatrix} K_- \\ K_+ \end{pmatrix} D(\alpha) = \begin{pmatrix} K_- + \alpha F_- + \frac{1}{8}\alpha^2 \\ K_+ + \alpha^* F_+ + \frac{1}{8}\alpha^{*2} \end{pmatrix} \tag{13}$$

$$S^\dagger(\beta)\begin{pmatrix} K_- \\ K_+ \end{pmatrix} S(\beta) = \begin{pmatrix} \cosh^2 r & e^{2i\theta}\sinh^2 r \\ e^{-2i\theta}\sinh^2 r & \cosh^2 r \end{pmatrix}\begin{pmatrix} K_- \\ K_+ \end{pmatrix} + \cosh r \sinh r \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}. \tag{14}$$

We now define CSs of the $osp(1/2, R)$ algebra as

$$|\alpha\beta\rangle = s(\beta)D(\alpha)|0\rangle. \tag{15}$$

Substituting (9) into (15) and using (7) and (8), one can obtain an explicit expression of these states in the number representation,

$$|\alpha\beta\rangle = \sum_{n=0}^{\infty}(n!\cosh r)^{-1/2}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2} \times \exp\left[-\frac{1}{8}(|\alpha|^2 - \alpha^2 e^{i\theta}\tanh r)\right]\cdot H_n\left[\frac{\alpha}{2}(e^{i\theta}\sinh r)^{-1/2}\right]|n\rangle. \tag{16}$$

From the definition (15) it can be seen that these CSs are normalized, i.e., $\langle\alpha\beta|\alpha\beta\rangle = 1$, but not orthogonal to each other. They have the following orthogonality relation:

$$\langle\alpha'\beta'|\alpha\beta\rangle = A^{1/2}(\beta,\beta')\exp\{-\tfrac{1}{8}(|\alpha|^2 + |\alpha'|^2) + \tfrac{1}{2}A(\beta,\beta')[B(\beta,\beta')\alpha^2 + 2\alpha\alpha'^* + B^*(\beta,\beta')\alpha^{*2}]\} \tag{17}$$

with

$$A(\beta,\beta') = (\cosh r \cosh r' - e^{i(\theta-\theta')} \sinh r \sinh r')^{-1} \tag{18}$$

$$B(\beta,\beta') = e^{-i\theta} \sinh r \cosh r' - e^{-i\theta'} \cosh r \sinh r'. \tag{19}$$

In the derivation of (17) we have used

$$\sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} H_n(x) H_n(y) = (1-t^2)^{-1/2} \exp\{(1-t^2)^{-1}[2xyt - (x^2+y^2)t]\}. \tag{20}$$

As is well known, the core of CSs is their completeness. It can be proved that these states defined in (15) form an overcomplete Hilbert space with the following completeness relation,

$$\int d^2\alpha\, d^2\beta\, \sigma(\alpha,\beta)|\alpha\beta\rangle\langle\alpha\beta| = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{21}$$

where the weight function $\sigma(\alpha,\beta)$ is given by

$$\sigma(\alpha,\beta) = \frac{1}{\pi}\delta(\mathrm{Re}\,\beta)\delta(\mathrm{Im}\,\beta). \tag{22}$$

Therefore these states defined in (15) satisfy the basic requirements [1] (i.e., continuity and overcompleteness) as CSs, they can be called $osp(1/2, R)$ CSs.

From (9) and (15), using (7) and (8) one can see that when $\beta = 0$, the $osp(1/2, R)$ CSs reduce to the well known Glauber CSs [7], when $\alpha = 0$, they reduce to CSs of the $su(1,1)$ algebra (corresponding to $k = 1/2$ and $1/4$ representations) [8]. Essentially, squeezed states in quantum optics [9] is the $osp(1/2, R)$ CSs. Therefore, the $osp(1/2, R)$ can encompass the Glauber CSs, $su(1,1)$ CSs and squeezed states within a common formalism.

It is obvious that the $osp(1/2, R)$ CSs defined in this paper are quite different from those CSs in [3,4]. The former contains two complex parameters α and β without Grassmann variables while the latter contains either a complex variable or a Grassmann variable. On the other hand, the latter has the property of being the minimum uncertainty states in the sense that they minimize the dispersion of the quadratic Casimir operator. It can be shown that the CSs defined in (15) are not eigenstates of the linearized Casimir operator of the $osp(1/2, R)$ algebra, $g^{ij}\langle X_i\rangle X_j$, where g^{ij} is the Cartan–Killing metric tensor and X_i denotes the $osp(1/2, R)$ generators; they cannot minimize the dispersion of the quadratic Casimir operator, $\Delta C_2 = g^{ij}\langle X_i X_j\rangle - g^{ij}\langle X_i\rangle X_j$, so that they are not minimum uncertainty states. However, it can be easily checked that they are eigenstates of the operator $\cosh r F_- + \sinh r F_+$,

$$(\cosh r F_- + \sinh r F_+|\alpha\beta\rangle = 2(\alpha\cosh r + \alpha^* e^{i\theta}\sinh r)|\alpha\beta\rangle. \tag{23}$$

It is well known that the D-algebras [10] for the CSs and the diagonal CS projectors are very useful for the laser theory [11] and the study of quantum spin systems [12, 13].

The D-algebra of an $osp(1/2, R)$ generator A for CSs (15) can be defined as

$$A|\alpha\beta\rangle \equiv D^k(A)|\alpha\beta\rangle \qquad \langle\alpha\beta|A \equiv D^b(A)\langle\alpha\beta| \qquad D^k(A) = [D^b(A)]^* \tag{24}$$

and the D-algebra on the CS projector $|\alpha\beta\rangle\langle\alpha\beta|$ can be defined as

$$A|\alpha\beta\rangle\langle\alpha\beta| \equiv D^l(A)|\alpha\beta\rangle\langle\alpha\beta| \qquad |\alpha\beta\rangle\langle\alpha\beta|A \equiv D^r(A)|\alpha\beta\rangle\langle\alpha\beta| \tag{25}$$

where we have $D^r(A) = (D^l(A))^*$. A theorem for the D-algebras can be found as follows:

$$AB|\alpha\beta\rangle = D^k(A)D^k(B)|\alpha\beta\rangle \qquad AB|\alpha\beta\rangle = D^l(A)D^l(B)|\alpha\beta\rangle. \tag{26}$$

It follows from (9) and (15) that

$$S(\beta)F_-D(\alpha)|0\rangle = \frac{1}{4}\alpha|\alpha\beta\rangle \qquad S(\beta)F_+D(\alpha)|0\rangle = \left(\frac{1}{8}\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\beta\rangle. \tag{27}$$

With the help of (11) and (12), the right-hand side of (27) can also be expressed as

$$S(\beta)F_-D(\alpha)|0\rangle = (\cosh r F_- - \mathrm{e}^{\mathrm{i}\theta}\sinh r F_+|\alpha\beta\rangle \tag{28}$$

$$S(\beta)F_+D(\alpha)|0\rangle = (\cosh r F_+ - \mathrm{e}^{\mathrm{i}\theta}\sinh r F_-|\alpha\beta\rangle. \tag{29}$$

By using (25)–(27), one can get

$$F_+|\alpha\beta\rangle = \left[\cosh r\left(\frac{1}{8}\alpha^* + \frac{\partial}{\partial\alpha}\right) + \frac{1}{4}\alpha\mathrm{e}^{-\mathrm{i}\theta}\sinh r\right]|\alpha\beta\rangle \tag{30}$$

$$F_-|\alpha\beta\rangle = \left[\frac{1}{4}\cosh r + \mathrm{e}^{\mathrm{i}\theta}\sinh r\left(\frac{1}{8}\alpha^* + \frac{\partial}{\partial\alpha}\right)\right]|\alpha\beta\rangle. \tag{31}$$

Then we obtain the ket D-algebras of generators F_+ and F_- as follows:

$$D^k(F_-) = \frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\theta}\sinh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right] \tag{32}$$

$$D^k(F_+) = \frac{1}{2}\left[\cosh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \alpha\mathrm{e}^{-\mathrm{i}\theta}\sinh r\right]. \tag{33}$$

Similarly, one can obtain ket D-algebras of other $osp(1/2, R)$ generators:

$$D^k(K_+) = \frac{1}{2}\left[\cosh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\mathrm{e}^{-\mathrm{i}\theta}\sinh r\right]^2 \tag{34}$$

$$D^k(K_-) = \frac{1}{2}\left[\mathrm{e}^{\mathrm{i}\theta}\sinh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right]^2 \tag{35}$$

$$D^k(K_0) = \frac{1}{2}\left[\cosh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\mathrm{e}^{-\mathrm{i}\theta}\sinh r\right] \times\left[\mathrm{e}^{\mathrm{i}\theta}\sinh r\left(\frac{1}{4}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right] + \frac{1}{4}. \tag{36}$$

The bra D-algebras can be given by the conjugation relation $D^b(A) = [D^k(A)]^*$.

It can be shown that these D operators of the $osp(1/2, R)$ generators in the CS space satisfy the same structure relations of the $osp(1/2, R)$ algebra:

$$[D^k(K_0), D^k(K_\pm)] = \pm D^k(K_\pm) \qquad [D^k(K_+), D^k(K_-)] = -2D^k(K_0) \tag{37}$$

$$\begin{gathered}[D^k(K_0), D^k(F_\pm)] = \pm\tfrac{1}{2}D^k(F_\pm) \qquad [D^k(K_\pm), D^k(F_\pm)] = 0 \\ [D^k(K_\pm), D^k(F_\mp)] = \mp D^k(F_\pm)\end{gathered} \tag{38}$$

$$\{D^k(F_\pm), D^k(F_\pm)\} = D^k(K_\pm) \qquad \{D^k(F_+), D^k(F_-)\} = D^k(F_0). \tag{39}$$

The ket D-operators also satisfy the similar commutation and anticommutation relations, and the bra D-operators commute with the ket D-operators, so that the D-algebra representations of the $osp(1/2, R)$ algebra are differential realizations of the algebra in the CS space.

On the other hand, the CS projector $|\alpha\beta\rangle\langle\alpha\beta|$ provides a basis in which most physically reasonable operators may be expanded. The left D-algebras on the projector $|\alpha\beta\rangle\langle\alpha\beta|$ can be found to be

$$D^l(F_-) = \frac{1}{2}\left[e^{i\theta}\sinh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right] \tag{40}$$

$$D^l(F_+) = \frac{1}{2}\left[\cosh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha e^{-i\theta}\sinh r\right] \tag{41}$$

$$D^l(K_-) = \frac{1}{2}\left[e^{i\theta}\sinh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right]^2 \tag{42}$$

$$D^l(K_+) = \frac{1}{2}\left[\cosh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha e^{-i\theta}\sinh r\right]^2 \tag{43}$$

$$\begin{aligned}D^l(K_0) = {} & \frac{1}{2}\left[\cosh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha e^{-i\theta}\sinh r\right] \\ & \times\left[e^{i\theta}\sinh r\left(\frac{1}{2}\alpha^* + 2\frac{\partial}{\partial\alpha}\right) + \frac{1}{2}\alpha\cosh r\right] + \frac{1}{4}.\end{aligned} \tag{44}$$

The right D-algebras can be obtained through $D^r(A) = [D^l(A)]^*$. It can be checked that these D-algebras on the projector $|\alpha\beta\rangle\langle\alpha\beta|$ satisfy the structure relations of the $osp(1/2, R)$ algebra. Therefore, the D-algebra representations on the CS projector also give a differential realization of the $osp(1/2, R)$ algebra.

In conclusion, we have presented a new kind of CSs for the Lie superalgebra $osp(1/2, R)$ and obtained the D-algebra representations of the superalgebra on the CS space and the CS projector space. It has been shown that these $osp(1/2, R)$ CSs can encompass the Glauber CSs, $su(1, 1)$ CSs and squeezed states within a common formalism. We hope these CSs and the D-algebras of the $osp(1/2, R)$ can find some applications in physics and mathematical physics. In our further work, we will give path integral formalism of these $osp(1/2, R)$ CSs and some applications of them in quantum physics.

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